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A simple inverse heat conduction method with optimization

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Abstract—A simple numerical method for one-dimensional nonlinear inverse heat conduction problem solving based on finite difference principle is presented. Time-step size is variable and chosen in such a way ensuring optimal results. Numerical stability analysis is derived and stochastic error contributions to the solution quality and inversely computed distance influence are demonstrated. Simulated experiments using inexact data illustrate usability of the method

INTRODUCTION

THE BOUNDARY inverse heat conduction problem (BIHCP) deals with determining surface temperature, heat flux or heat transfer coefficient history from temperature readings inside the body. Numerous methods for the solution of such problems have been developed, many of them being based on the methods of Beck *et al.* [1] with least squares minimization and future temperatures utilization. There is another approach, namely, that of finite difference solution of Alifanov [2] while Backus and Gilbert [3] developed a method for qualitative sensitivity computing and determining solution exactness estimate. Hills *et al.* [4] used the latter method for 2D slab BIHCP solution. Alifanov *et al.* [5] describe the 'iterative regularization' method of ill-posed problems solution. An adaptive sequential method as a generalization of Beck's function specification method is developed in the paper of Flach and Özişik [6].

In the present paper, the optimization of finite difference solution with time-step estimation for time-dependent surface conditions and temperature-dependent properties is investigated. First, the stability analysis of solution is developed, then the effectiveness of time step control is demonstrated using temperature data measurement simulation.

PROBLEM FORMULATION

Governing equations of a *direct* heat conduction problem in material with constant density are as follows:

$$\rho c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right)$$

$$\text{for} \quad a \leqslant x \leqslant b \quad \text{and} \quad 0 \leqslant t \leqslant t_c \qquad (1)$$

$$T(x,0) = T_0(x) \quad \text{for} \quad a \leqslant x \leqslant b \qquad (1a)$$

$$\beta_1 \frac{\partial T(a,t)}{\partial x} + \gamma_1 T(a,t) = f_1(t),$$

$$\beta_2 \frac{\partial T(b,t)}{\partial x} + \gamma_2 T(b,t) = f_2(t),$$

$$\text{for} \quad 0 < t \leqslant t_c. \qquad (1b, c)$$

While in the BIHCP formulation, temperature-dependent thermal properties, initial temperature and additional conditions (temperature readings)

$$T(c,t) = f_3(t), \quad T(d,t) = f_4(t)$$

$$\text{for} \quad 0 < t \leqslant t_c \qquad (2a, b)$$

are known and surface conditions f_1, f_2, equations (1b, c) are to be solved (see Fig. 1).

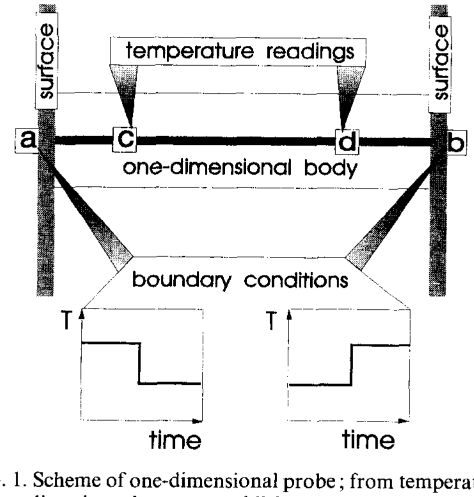

FIG. 1. Scheme of one-dimensional probe; from temperature readings in c, d we can establish temperatures in a, b.

NOMENCLATURE

$c(T)$	specific heat [J kg^{-1} K^{-1}]
$E_{e,i}$	euclidean norms
$f_{1,2,3,4}$	known functions [K]
Fo	Fourier number [−]
$k(T)$	thermal conductivity [W m^{-1} K^{-1}]
l	number of time divisions [−]
L	length [m]
m	time division [−]
t	time [s]
T	temperature [°C]
x	spatial coordinate [m].

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
$\beta_{1,2}, \gamma_{1,2}$	constants determining type of boundary conditions
δ	relative inversely computed distance [%]
ε	small number, stochastic error
λ	eigenvalues
ρ	mass density.

Subscripts

c	end time
e	exact solution
i	spatial index
i	inverse solution
0	initial temperature.

Superscripts

$-$	temperatures from preceding time
k	iteration index
$2^{(j-1)\Delta t_m}$	time step index.

The BIHCP is solved as an initial value problem. The computing domain is divided into internal interval $\langle c, d\rangle$ and two external intervals $\langle a, c)$, $(d, b\rangle$. Equation (1) is discretized with implicit formulation of finite difference method:

$$\rho c(\bar{T}_i)\frac{T_i - T_i^-}{\Delta t} = \frac{1}{\Delta x}\left(k_{i+1/2}\frac{T_{i+1} - T_i}{\Delta x} + k_{i-1/2}\frac{T_{i-1} - T_i}{\Delta x}\right)$$

where

$$i = c+1, \ldots, d-1, \quad \bar{T}_i = \frac{T_i + T_i^-}{2}$$

and

$$k_{i\pm 1/2} = k\left(\frac{T_i + T_{i\pm 1}}{2}\right). \tag{3}$$

Temperatures T_i in interval $\langle c, d\rangle$—direct heat conduction problem—are calculated from equations (3) with the use of traditional matrix solver and from known initial condition $T(x, 0)$ for $c < x < d$ and boundary conditions $T(c, t)$, $T(d, t)$ for $0 < t \leqslant t_c$.

Temperatures in intervals $\langle a, c)$ and $(d, b\rangle$—inversely determined temperatures—are computed from equation (3), where we independently evaluate T_{i-1} and T_{i+1}

$$T_{i-1} = \left(1 + \frac{k_{i+1/2}}{k_{i-1/2}} + \rho\frac{c(\bar{T}_i)(\Delta x)^2}{k_{i-1/2}\Delta t}\right)T_i - \frac{k_{i+1/2}}{k_{i-1/2}}T_{i+1} - \rho\frac{c(\bar{T}_i)(\Delta x)^2}{k_{i-1/2}\Delta t}T_i^- \tag{4}$$

$$T_{i+1} = \left(1 + \frac{k_{i-1/2}}{k_{i+1/2}} + \rho\frac{c(\bar{T}_i)(\Delta x)^2}{k_{i+1/2}\Delta t}\right)T_i - \frac{k_{i-1/2}}{k_{i+1/2}}T_{i-1} - \rho\frac{c(\bar{T}_i)(\Delta x)^2}{k_{i+1/2}\Delta t}T_i^- \tag{5}$$

and solve in $\langle a, c)$ and $(d, b\rangle$ step by step (so-called space marching method). Equations (4), (5) are solved iteratively due to the non-constancy in $k(T)$ and $c(T)$; the end of iteration process being controlled by norm

$$\max_i |T_i^{k+1} - T_i^k| < \varepsilon.$$

From these resolved temperatures, inside and on the surface of the body, we can reconstruct boundary conditions also for $\beta_{1,2} \neq 0$ (in equations (1b, c))—it means solving the inverse problem for heat flux or heat transfer coefficient.

STABILITY AND UNIQUENESS

Alifanov [2] used the above-described method for the linear stationary heat conduction equation without any stability analysis. Stability determination for nonlinear equations (4), (5) is not known, but linear equation stability analysis is possible.

Instead of (1), a linear heat conduction equation is established

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}. \tag{6}$$

With $g = 1/Fo = \Delta x^2/(\alpha\Delta t)$ it is possible to write equation (4) as

$$T_{i-1} - (2+g)T_i + T_{i+1} = gT_i^-. \tag{7}$$

The exact solution of (7) from Berezin and Zhidkov [7] has the form

$$T_i = c_1\lambda_1^i + c_2\lambda_2^i + p^i \tag{8}$$

where $c_1\lambda_1^i + c_2\lambda_2^i$ is a general solution of homogeneous equation, and p^i is the particular solution of equation (7). Eigenvalues λ_1, λ_2 can be obtained from the solution of characteristic equation

$$\lambda^2 - (2+g)\lambda + 1 = 0. \tag{9}$$

Solution of homogeneous equation is stable, if $\max(|\lambda_1|, |\lambda_2|) \leqslant 1$ and $\lambda_1 \neq \lambda_2$, if $\lambda_1 = \lambda_2$ the stability condition has the form $\max(|\lambda_1|, |\lambda_2|) < 1$; therefore from equation (9) it can be seen that numerical process is unstable for any value of g and the particular part of solution

$$p_i = -\rho A^{-1} T_i^-, \quad A = \begin{bmatrix} -(2+g) & 1 & & & \\ 1 & -(2+g) & 1 & & \\ & \cdots & & & \\ & & 1 & -(2+g) & 1 \\ & & & 1 & -(2+g) \end{bmatrix} \tag{10}$$

does not stabilize the process.

From the physical point of view, it is possible to say that the direct heat conduction solution attenuates high frequencies in boundary conditions more quickly than the low ones while the inverse heat conduction solution amplifies high-frequency components in the signal and, for this reason, it is not possible to use BIHCP methods with very noisy experimental data.

There are other problems connected with the uniqueness of boundary condition history reconstruction. It is possible to obtain solution only with a certain degree of approximation and this is why use of BIHCP in practice requires precise input (and from the view of stability also output) analysis.

NUMERICAL ALGORITHM

We know:

$$T(c,t) = T^{2^{(j-1)}\Delta t_m}(c,t)$$

$$T(d,t) = T^{2^{(j-1)}\Delta t_m}(d,t)$$

$$\text{for } j = 1,\ldots,l \text{ and } 0 < t \leqslant t_c$$

where Δt_m is a minimal (basic) time step (time step of measurement) and every subsequent time step double the preceding one, l is a number of different time steps. Further we know L, c, d, t_c, $c(T)$, ρ and $k(T)$.

We seek:

$T(a,t)$, $T(b,t)$ with the time step $2^{(j-1)}\Delta t_m$ (time t_c division $m = t_c/2^{(j-1)}\Delta t_m$) for minimal error of solution.

Realization:

(i) for the constant Δx and with the changing (doubling) time-step we solve temperatures inside the interval (c,d) from equation (3);

(ii) in external fields $\langle a,c)$ and $(d,b\rangle$ we use equations (4) and (5) respectively. For the different j ($j = 1,\ldots,l$) we evaluate norm

$$E_i = \| T^{2^j\Delta t_m}(a,t) - T^{2^{(j-1)}\Delta t_m}(a,t) \|. \tag{11}$$

We are watching for a minimum of this norm E_i of two successive solutions (with different time steps) and choose $m = t_c/2^{(j-1)}\Delta t_m$ for the norm minimum. Existence of a minimum ensures the optimal time-step (time-division) of the BIHCP solution.

Norm $E_e = \| T(a,t) - T_i(a,t) \|$ of exact (prescribed) data in a, $T(a,t)$ and inversely computed temperatures $T_i(a,t)$ is evaluated for comparison.

NUMERICAL EXPERIMENT CONDITIONS

For the representative test cases copper, as material of computed one-dimensional body with thermophysical properties $\rho = 8930$ kg m^{-3}, $k(T) = 386.7 - 0.0078T$ W m^{-1} K^{-1}, $c(T) = 377.26 + 0.136T$ J kg^{-1} K^{-1} and length $L = 0.04$ m, is considered. Time of measurement was chosen as $t_c = 3.2$ s, the initial condition is $T(x,0) = 0$°C and boundary conditions simulating measured temperatures for different cases are shown in Fig. 2.

The parameters of finite difference schema (3) were: space division of (a,b) $n = 80$, time t_c division $m = 1280$. Such a solution simulates the measurement without stochastic error (noise). Noise is subsequently generated by randomizer with the amplitude of 0.1%, 1% and 2% of signal value.

For the inverse conduction solution we choose $n = 40$, various $m = \{5, 10, 20, 40, 80, 160, 320, 640\}$ ($\Delta t_m = 0.005$ s), while additional conditions (temperature readings in c, d) are taken from the above-described direct solution on (a,b). The distance from the surface (i.e. the length of (a,c) or (d,b)) equals 10% of the total length L.

RESULTS

First the stability of inverse reconstruction of boundary condition No. 1 (Fig. 2) with exact data-

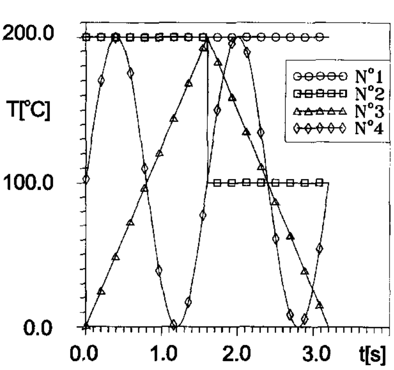

FIG. 2. Boundary conditions (1—steady, 2—step, 3—sinus and 4—pyramid-like) for test cases.

readings is numerically investigated. Time development of surface temperature $T(a, t)$ reconstruction for the two different time divisions m is shown in Fig. 3(a). Phase trajectories for exact data and different m are shown in Fig. 3(b) and for inexact data with stochastic error $\varepsilon \leqslant |0.01\%|$ in Fig. 3(c). It follows that the solution is stable and stability is expected even if inexact data with larger error were used.

The first of the test cases is a reconstruction of boundary conditions No. 2 (Fig. 2) from exact data shown in Fig. 4(a). The process of optimal time-step choosing is illustrated in Fig. 4(b). There are norms E_c, E_i for different time-interval (t_c) divisions m. For the first case an optimum in both norms is for $m = 320$. The situation for inexact data with stochastic error $|\varepsilon| \leqslant 0.1\%$ and $|\varepsilon| \leqslant 1.0\%$, $|\varepsilon| \leqslant 2.0\%$ is depicted in Figs. 5(a) and 6(a) respectively. Optimal m for these solutions are, according to Figs. 5(b), 6(b), $m = 80$ and 40, respectively.

The influence of distance of inverse temperature reconstruction on a quality of solution with $|\varepsilon| \leqslant 0.1\%$ is shown by comparison of Figs. 5(a), (b) ($\delta = 10\%$ of L) the optimal time division being $m = 80$ and Figs. 7(a), (b) ($\delta = 20\%$ of L), $m = 40$.

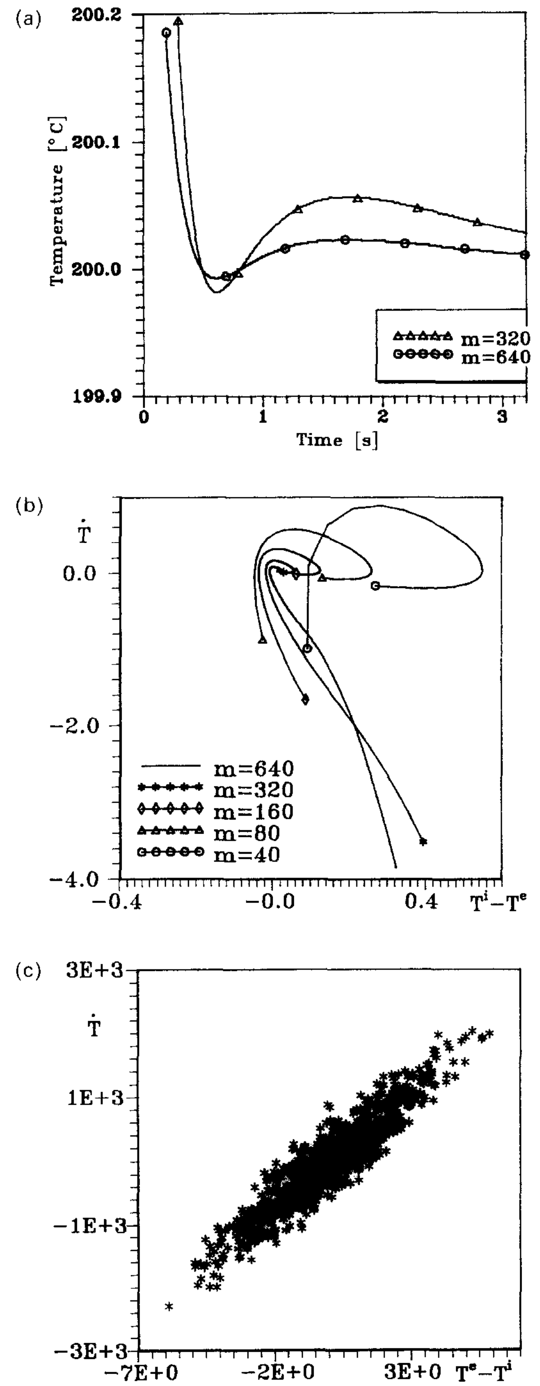

FIG. 3. (a) Estimated surface temperature for steady boundary condition and for exact data, $\delta = 10\%$. (b) Phase trajectories of solution for different time divisions and for steady boundary condition and exact data, $\delta = 10\%$. (c) Inverse solution in phase space for steady condition with inexact data, $|\varepsilon| \leqslant 0.01\%$, $m = 640$, $\delta = 10\%$.

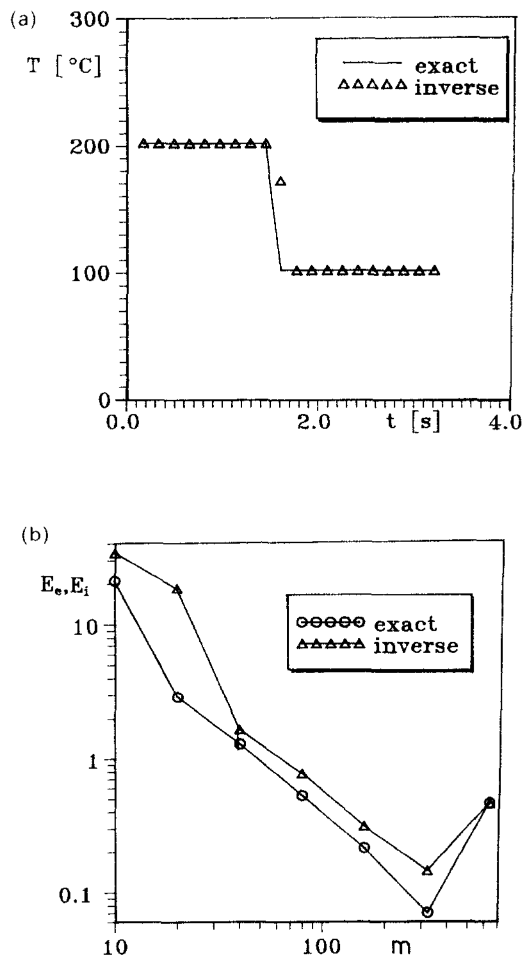

FIG. 4. (a) Estimation of step surface temperature history, stochastic error $|\varepsilon| = 0\%$, optimal time division $m = 320$, $\delta = 10\%$. (b) Norms E_c, E_i for test case 4(a) with minimum for $m = 320$.

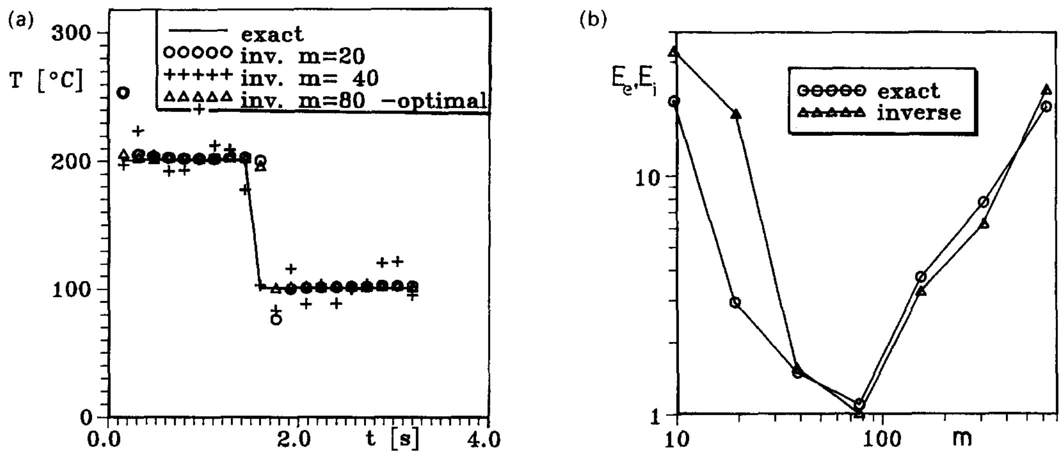

FIG. 5. (a) Estimation of step surface temperature history, $|\varepsilon| \leqslant 0.1\%$, $\delta = 10\%$, optimum $m = 80$. (b) Norms E_e, E_i for test case 5(a) with minimum for $m = 80$.

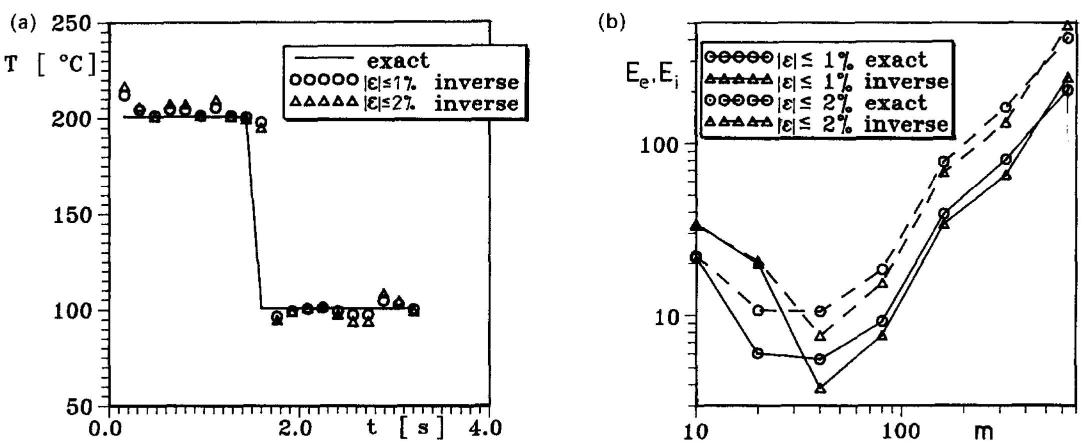

FIG. 6. (a) Estimation of step surface temperature history for $|\varepsilon| \leqslant 1.0\%$ and $|\varepsilon| \leqslant 2\%$, respectively, $\delta = 10\%$, optimum $m = 40$. (b) Norms E_e, E_i for test cases 6(a) with minimum $m = 40$.

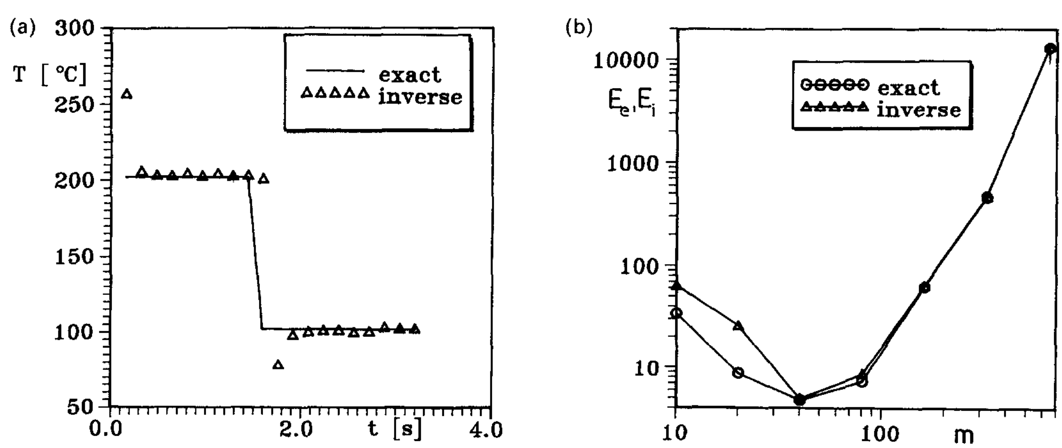

FIG. 7. (a) Estimation of step surface temperature history for $|\varepsilon| \leqslant 0.1\%$, $\delta = 20\%$, optimum $m = 40$. (b) Norms E_e, E_i for test case 7(a) with minimum $m = 40$.

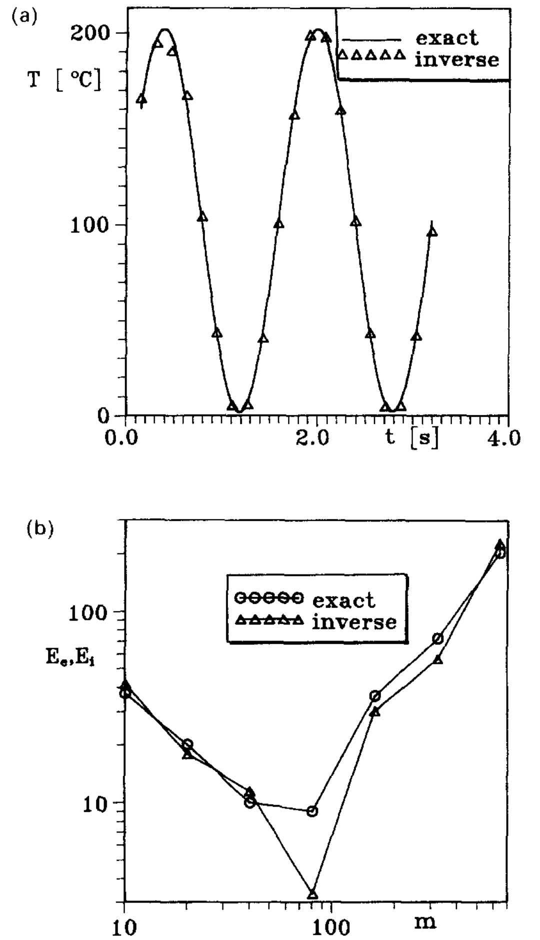

FIG. 8. (a) Estimation of sinus surface temperature history for $|\varepsilon| \leqslant 1.0\%$, $\delta = 10\%$, optimum $m = 80$. (b) Norms E_e, E_i for test case 8(a) with minimum $m = 80$.

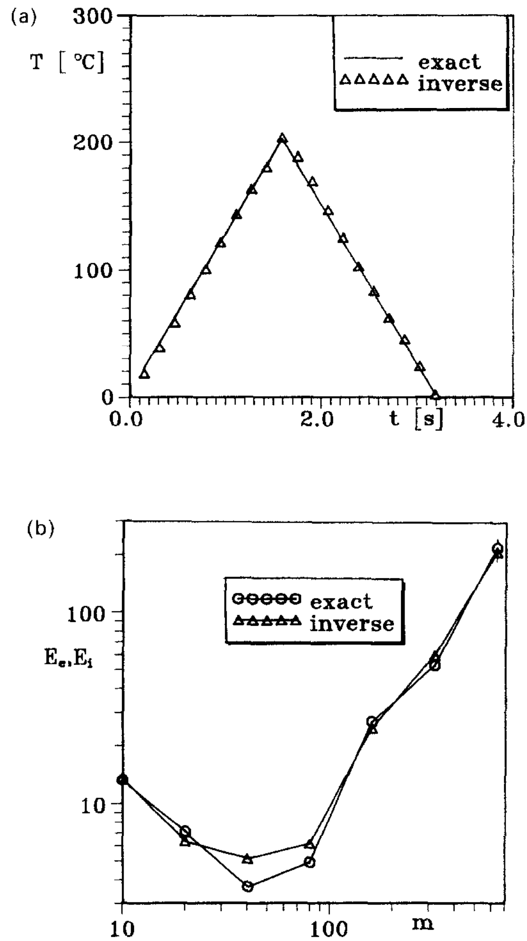

FIG. 9. (a) Estimation of pyramid surface temperature history for $|\varepsilon| \leqslant 1.0\%$, $\delta = 10\%$, optimum $m = 40$. (b) Norms E_e, E_i for test case 9(a) with minimum $m = 40$.

Reconstruction of sinus-like b.c. (Fig. 2, No. 3), with $|\varepsilon| \leqslant 1.0\%$ and $\delta = 10\%$ of L, is on Figs. 8(a), (b) with optimum at $m = 80$. The results of computation for the same conditions (only with pyramid-like temperature history) on the surface (Fig. 2, No. 4) are illustrated in Figs. 9(a), (b).

CONCLUSIONS

A new variable time step method has been presented for the one-dimensional inverse heat conduction problem. The advantage of this algorithm is seen in its simplicity, computational efficiency and in a good ability to compute from data with small-amplitude (up to 2% of signal value) and high-frequency error (stochastic noise).

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